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# Aspects of the free field description of string theory on $AdS_3$

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Abstract: The near boundary limit of string theory in $AdS_3$ is analysed using the Wakimoto free field representation of $\mathrm{SL}(2,\mathbb{R})$. The theory is considered as a direct product of the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset and a free boson. Correlation functions are constructed generalizing to the non-compact case the integral representation of conformal blocks introduced by Dotsenko in the compact SU(2) CFT. Sectors of the theory obtained by spectral flow manifestly appear. The formalism naturally leads to consider scattering processes violating winding number conservation. The consistency of the procedure is verified in the factorization limit.

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## 1. Introduction

There are many motivations to study string theory in three dimensional Anti de Sitter spacetime. It was realized more than one decade ago that an $AdS_3$ metric plus antisymmetric tensor field provide an exact solution to the consistency conditions for string propagation in non-trivial background fields. The corresponding $\sigma$-model is a WZW model on the $\mathrm{SL}(2,\mathbb{R})$ group manifold (or on $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2) = H_3^+$ if one considers the euclidean version). More recently, the AdS/CFT duality conjecture supplied an additional motivation. Since both sides of the duality map, the three dimensional string theory and the two dimensional CFT, are in principle completely solvable, this toy model raises the hope to explicitly work out the details of the conjecture at the string level.

However, even though the theory has undergone a thorough examination over the last years, many important questions are still unanswered. In particular, it is not yet clear what the spectrum of the theory is. The prescription to consider the principal continuous series and the discrete representations (lowest and highest weight) of $\mathrm{SL}(2,\mathbb{R})$ (or its universal covering, $\widetilde{\mathrm{SL}(2,\mathbb{R})}$) with the spin $j$ bounded by unitarity leads to an unnatural limit on the level of excitation of the string states and to a partition function which is not modular invariant (for a review and a complete list of references see [1]), and it fails in the interacting theory [2].

An interesting proposal was recently advanced by Maldacena and Ooguri [3]. They realized that the $\mathrm{SL}(2,\mathbb{R})$ WZW model has a spectral flow symmetry which originates new admissible representations for the string spectrum. Taking them into



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account the problems mentioned above do not arise and it is possible to consistently keep the bound on the spin $j$ to avoid negative norm states in the free theory. This approach seems promising and it would be very important to complete it by considering interactions. In fact, a consistent string theory should provide a mechanism to avoid ghosts at the interacting level, i.e. non unitary states should decouple in physical processes. But the computation of correlation functions in this model presents several difficulties due to the non-compact nature of $H_3^+$, which renders the proof of unitarity of the full theory highly non trivial. Various attempts to include interactions have been developed in recent years using different methods. Certain correlators of the $\mathrm{SL}(2,\mathbb{R})$ WZW model have been computed by functional integration in [4]. The bootstrap formalism was implemented in [5] and two- and three-point functions for arbitrary spin $j$ were recently computed in [6] using the path integral approach. The computation of higher point functions is important to completely establish the consistency of the theory, but it gives rise to technical obstacles and complete expresions are not yet available.

Until more efficient calculational methods emerge, the free field approach provides a useful tool to obtain some information. It is suitable for describing processes in the near boundary region of $AdS_3$ (though results in [6] suggest that it could apply to a larger region). The approach was used in [2] to study the factorization limit of $N$-point functions in the $H_3^+$ WZW model and determine the unitarity of the theory. It is the purpose of this article to extend the free field formalism to manifestly include the spectral flow symmetry. A direct extension of Dotsenko's method to compute the conformal blocks in the compact SU(2) CFT [7] to the non-compact SL(2) (or $H_3^+$) group manifold is found adequate to deal with the spectral flow symmetry in vertex operators and scattering processes and to describe interactions either conserving or violating winding number conservation. In fact, the spectral flow parameter $\omega$ is identified with the winding number of the string in $AdS_3$ and, as explained in references [3, 8], it does not need to be conserved by interactions.

The general method carried out in the following sections to construct the theory goes along the steps pursued in the proof of the no-ghost theorem [9, 10]. It begins with the $H_3^+$ WZW model. Since the minus sign in the norm of some states of the theory can be traced to the U(1) part of the current algebra, the states created by the moments of this current are removed by considering the coset $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$. Finally, string theory in $AdS_3$ is recovered by taking the tensor product of the coset with the state space of a timelike free boson.

The paper is organized as follows. In section 2 the free field description of SU(2) CFT is reviewed by directly extending it to the non-compact case. The integral representation of the conformal blocks and the mechanism to find the charge asymmetry conditions leading to non-vanishing correlators is recalled. In section 3 the quotient of $\mathrm{SL}(2,\mathbb{R})$ by U(1) is considered along the same path. The formalism naturally leads to find new expressions for the vertex operators and new sets of charge asymmetry



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conditions. This lays the ground to manifestly introduce the spectral flow symmetry into string theory on $AdS_3$ in section 4, similarly as what is done in the compact case [11]. The scattering amplitudes for physical states are considered in section 5 and their factorization properties are analysed in order to check the consistency of the procedure. The vertex operators introduced in section 3 are found useful to describe processes violating winding number conservation. Finally the conclusions can be found in section 6.

## 2. Review of the free field representation of CFT

In this section we review Dotsenko's construction of the free field representation of SU(2) conformal field theory [7] by extending it directly to the SL(2) non-compact case.

The Wakimoto representation of SL(2) current algebra [12] is realized by three fields $\beta, \gamma, \phi$, with correlators given by

$$\langle \beta(z)\gamma(w)\rangle = \frac{1}{z-w}\,, \qquad \langle \phi(z)\phi(w)\rangle = -\ln(z-w)\,. \tag{2.1}$$

There are also $\bar z$ dependent antiholomorphic fields ($\bar\beta(\bar z)$, $\bar\gamma(\bar z)$, $\phi(\bar z)$). However we shall discuss the left moving part of the theory only and assume that all the steps go through to the right moving part as well, indicating the left-right matching conditions where necessary.

The SL(2) currents are represented as

$$\begin{aligned} J^+(z) &= \beta \\ J^3(z) &= -\beta\gamma - \frac{\alpha_+}{2}\partial\phi \\ J^-(z) &= \beta\gamma^2 + \alpha_+\gamma\partial\phi + k\partial\gamma\,, \end{aligned} \tag{2.2}$$

where $\alpha_+ = \sqrt{2(k-2)}$ and $k$ is the level of the SL(2) algebra. They verify the following operator algebra

$$\begin{aligned} J^+(z)J^-(w) &= \frac{k}{(z-w)^2} - \frac{2}{(z-w)}J^3(w) + \mathrm{RT} \\ J^3(z)J^\pm(w) &= \pm\frac{1}{(z-w)}J^\pm(w) + \mathrm{RT} \\ J^3(z)J^3(w) &= \frac{-k/2}{(z-w)^2} + \mathrm{RT}\,. \end{aligned} \tag{2.3}$$

Expanding in Laurent series

$$J^a(z) = \sum_{n=-\infty}^{\infty} J^a_n\, z^{-n-1} \tag{2.4}$$

the coefficients $J_n^a$ satisfy a Kac-Moody algebra given by

$$[J_n^a, J_m^b] = i\epsilon_c^{ab} J_{n+m}^c - \frac{k}{2}\eta^{ab} n\delta_{n+m,0}\,, \tag{2.5}$$

where the Cartan-Killing metric is $\eta^{ab} = \mathrm{diag}(1,1,-1)$ and $\epsilon_c^{ab}$ is the Levi-Civita antisymmetric tensor.

The Sugawara stress-energy tensor is

$$T_{\mathrm{SL}(2)}(z) = \beta\partial\gamma - \frac{1}{2}\partial\phi\partial\phi - \frac{1}{\alpha_+}\partial^2\phi \tag{2.6}$$

and it leads to the following central charge of the Virasoro algebra

$$c = 3 + \frac{12}{\alpha_+^2} = \frac{3k}{k-2}\,. \tag{2.7}$$

The primary fields of the SL(2) conformal theory $\Phi_m^j(z)$ satisfy the following OPE with the currents

$$\begin{aligned} J^+(z)\Phi_m^j(w) &= \frac{(j-m)}{z-w}\Phi_{m+1}^j(w) + \mathrm{RT} \\ J^3(z)\Phi_m^j(w) &= \frac{m}{z-w}\Phi_m^j(w) + \mathrm{RT} \\ J^-(z)\Phi_m^j(w) &= \frac{(-j-m)}{z-w}\Phi_{m-1}^j(w) + \mathrm{RT}\,. \end{aligned} \tag{2.8}$$

The corresponding vertex operators can be expressed as [14]

$$\Phi_m^j(z) = \gamma^{j-m} e^{\frac{2j}{\alpha_+}\phi} \tag{2.9}$$

and their conformal dimensions are

$$\Delta(\Phi_m^j) = -\frac{j(j+1)}{k-2}\,. \tag{2.10}$$

The next object of the free field realization is the screening operator. It has to commute with all the currents, i.e. it should have no singular terms in the OPE with them. Up to a total derivative this is satisfied by the operators [13, 14]

$$S_+(z) = \beta(z) e^{-\frac{2}{\alpha_+}\phi}\,; \qquad S_-(z) = \beta^{k-2} e^{-\alpha_+\phi}\,. \tag{2.11}$$

It can be checked that

$$\begin{aligned} J^+(z)S_\pm(w) &= \mathrm{RT}\,, \qquad J^3(z)S_\pm(w) = \mathrm{RT} \\ J^-(z)S_+(w) &= (k-2)\partial_w\left(\frac{e^{-\frac{2}{\alpha_+}\phi}}{z-w}\right) + \mathrm{RT} \\ J^-(z)S_-(w) &= (k-2)\partial_w\left(\frac{\beta^{k-3}e^{-\alpha_+\phi}}{z-w}\right) + \mathrm{RT}\,. \end{aligned} \tag{2.12}$$

The total derivatives do not contribute if one integrates $S_\pm$ over a closed contour. Then the screening operators

$$\mathcal{S}_\pm = \int_\mathcal{C} dz S_\pm(z) \tag{2.13}$$

commute with the current algebra, they have zero conformal weight and can be used inside correlation functions without modifying their conformal properties.

As shown by Dotsenko [7], to construct the integral representation for the conformal blocks one needs a conjugate operator for the fields $\Phi^j_m$ to avoid redundant contour integrations which render the representation incomplete. In order to find it, it is important to construct the operator conjugate to the identity, which determines the charge asymmetry conditions of the expectation values in the radial-type quantization of the theory. It has to commute with the currents and have conformal dimension zero. It is found to be

$$\tilde{\mathcal{I}}_0(z) = \beta^{k-1} e^{\frac{2(1-k)}{\alpha_+}\phi} . \tag{2.14}$$

Similarly as in the SU(2) case one finds that there is no double pole in the OPE $J^-(z)\tilde{\mathcal{I}}_0(w)$ and that the residue of the single pole is a spurious state which decouples in the conformal blocks for physical states.

The conjugate identity operator requires that the charge asymmetry in expectation values be

$$N_\beta - N_\gamma = k - 1 , \qquad \sum_i \alpha_i = \frac{2-2k}{\alpha_+} , \tag{2.15}$$

where $N_\beta(N_\gamma)$ refers to the number of $\beta(\gamma)$ fields in the correlator and $\alpha_i$ refers to the "*charge*" of $\phi(z_i)$. Strong remarks against attributing the charge asymmetry to the presence of the background charge operator in the expectation values are given by Dotsenko [7].

One can now construct the conjugate representation for the highest weight operators which turns out to be

$$\tilde{\Phi}^j_j(z) = \beta^{2j+k-1} e^{-\frac{2(j-1+k)}{\alpha_+}\phi} . \tag{2.16}$$

It can be checked that it satisfies the relations (2.8) corresponding to a highest weight field (i.e. $j = m$) and that its conformal dimension is (2.10). Furthermore, it can be shown that the two-point functions $\langle \tilde{\Phi}^j_j \Phi^j_{-j} \rangle$ do not require screening operators to satisfy the charge asymmetry conditions (2.15).

The naive prescription to compute the conformal blocks of the four-point functions, a straightforward generalization of the compact case, is

$$\left\langle \Phi^{j_1}_{m_1}(z_1)\Phi^{j_2}_{m_2}(z_2)\Phi^{j_3}_{m_3}(z_3)\tilde{\Phi}^{j_4}_{j_4}(z_4) \prod_i \mathcal{S}_+(u_i) \prod_j \mathcal{S}_-(v_j) \right\rangle , \tag{2.17}$$

where the number of screening operators has to be chosen according to the charge asymmetry conditions (2.15). Notice that it is possible to satisfy them using only one type of screening operators, namely $\mathcal{S}_+$. In the compact SU(2) case it seems convenient to use the conjugate representation operator in the highest weight position for computation of conformal blocks and correlation functions [7] since the other operators of the multiplet, $\tilde{\Phi}^j_m$, take more complicated expressions.

Conformal field theory based on $\widehat{\mathrm{SL}(2)}_k$ has been studied for fractional levels of $k$ and spins in [15, 16, 17]. Several technical difficulties arise from the occurrence of fractional powers of $\beta, \gamma$ fields. For applications to string theory in $AdS_3$ one needs to consider real values of the level $k$ satisfying $3 < c = \frac{3k}{k-2} \leq 26$ (depending on the internal space). The spin $j$ is determined by the mass shell and unitarity conditions. Let us briefly review this theory.

$AdS_3$ is the universal covering of the $\mathrm{SL}(2,\mathbb{R})$ group manifold ($\widetilde{\mathrm{SL}(2,\mathbb{R})}$). The sigma model describing string propagation in this background plus an antisymmetric tensor field is a WZW model. A well defined path integral formulation of the theory requires to consider an euclidean $AdS_3$ target space which is the $\mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2) = H_3^+$ group manifold. Using the Gauss parametrization, the WZW model can be written as

$$S = k \int d^2z \left[\partial\phi\bar{\partial}\phi + \bar{\partial}\gamma\partial\bar{\gamma}e^{2\phi}\right], \tag{2.18}$$

which describes strings propagating in three dimensional Anti-de Sitter space with curvature $-2/k$, euclidean metric

$$ds^2 = kd\phi^2 + ke^{2\phi}d\gamma d\bar{\gamma} \tag{2.19}$$

and background antisymmetric field

$$B = ke^{2\phi}d\gamma \wedge d\bar{\gamma}\,. \tag{2.20}$$

The boundary of $AdS_3$ is located at $\phi \to \infty$. Near this region quantum effects can be treated perturbatively, the exponent in the last term in (2.18) is renormalized and a linear dilaton in $\phi$ is generated. Adding auxiliary fields $(\beta, \bar{\beta})$ and rescaling, the action becomes [18]

$$S = \frac{1}{4\pi}\int d^2z \left[\partial\phi\bar{\partial}\phi - \frac{2}{\alpha_+}R\phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} - \beta\bar{\beta}e^{-\frac{2}{\alpha_+}\phi}\right]. \tag{2.21}$$

This description of the theory can be trusted for large values of $\phi$. Note that the last term in (2.21) is one of the screening operators (2.11). It is known from the free field representation of the minimal models [19] that the original Feigin-Fuchs prescription with contour integrals of screening operators is equivalent to the one with the screening charges in the action. It is usually assumed that the same equivalence holds in this model [18, 20].

The string states must be in unitary representations of $\mathrm{SL}(2,\mathbb{R})$ and satisfy the Virasoro constraints, $L_m|\Psi\rangle = 0$, $m > 0$ and $L_o|\Psi\rangle = |\Psi\rangle$. The last one implies

$$-\frac{j(j+1)}{(k-2)} + L = 1 \tag{2.22}$$

at excited level $L$. Notice that this expression is invariant under $j \to -j-1$.

Taking into account that the Casimir plays the role of mass squared operator, the mass spectrum of the theory is

$$M^2 = \frac{(L-1)}{2}\alpha_+^2\,. \tag{2.23}$$

Therefore, the ground state of the bosonic theory is a tachyon, the first excited level contains massless states and there is an infinite tower of massive states. If there is an internal compact space $\mathcal{N}$, eq. (2.22) becomes

$$-\frac{j(j+1)}{k-2} + L + h = 1\,, \tag{2.24}$$

where $h$ is the contribution of the internal part.

Unlike string theory in Minkowski spacetime, the Virasoro constraints do not decouple all the negative norm states. Since the physical spectrum of string theory is expected to be unitary, the admissible $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ representations are restricted. Only the following unitary series at the base are relevant (see [3]):

i) Principal discrete highest weight representations:

$$\mathcal{D}_j^- = \{|j,m\rangle; m = j, j-1, j-2, \ldots\}\,, \tag{2.25}$$

where $J_0^+|j,j\rangle = 0$.

ii) Principal discrete lowest weight representations:

$$\mathcal{D}_j^+ = \{|j,m\rangle; m = -j, -j-1, -j-2, \ldots\}\,, \tag{2.26}$$

where $J_0^-|j,-j\rangle = 0$.

Unitarity requires $j \in \mathbb{R}$ and $-1/2 < j < (k-2)/2$ for both discrete series.

iii) Principal continuous representation

$$\mathcal{C}_j = \left\{|j,m\rangle; j = -\frac{1}{2} + i\lambda; \lambda, m \in \mathbb{R}\right\}. \tag{2.27}$$

The full representation space is generated by acting on the states in these series with $J_n^{\pm,3}$, $n < 0$, and the corresponding representations are denoted by $\hat{\mathcal{D}}_j^{\pm}, \hat{\mathcal{C}}_j$.

Correspondingly the conformal blocks that are relevant for string theory involve arbitrary (in general complex or not positive integer) powers of the ghost fields, and it is not obvious how to deal with them in explicit calculations.

However other conceptual physical problems are faced when one tries to identify this model with a string theory. On the one hand, the bound on the spin of the discrete representations implies an unnatural bound on the level of excitation of the string spectrum. Moreover, modular transformations of the partition function of the theory defined in this way reintroduce states that are eliminated by the bound (see reference [1] for a complete discussion of these issues). Finally, interactions reintroduce, in the intermediate channels, the negative norm states that were eliminated by the unitarity bound [2].

A natural solution to these problems was recently proposed by Maldacena and Ooguri [3]. They noticed an extra symmetry in this theory which allows to consider other representations of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ in addition to those mentioned above. In order to extend the formalism reviewed in this section to that case, it is convenient to first discuss the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset theory.

## 3. Extension to the $\mathrm{SL}(2)/\mathrm{U}(1)$ coset CFT

The minus sign appearing in the norm of some states in representations of $\mathrm{SL}(2,\mathbb{R})$ is associated with the $\mathrm{U}(1)$ part of the current algebra. Therefore a unitary module can be obtained by removing all the states created by the moments of $J^3$. This procedure defines a module for the coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ and provides the basis for the proof of the no-ghost theorem for string theory in $AdS_3$ [9, 10]. In fact, the theorem is proved by first showing that all the solutions to the Virasoro constraints on physical states can be expressed as states in the $\mathrm{SL}(2)/\mathrm{U}(1)$ coset. Thus, string theory in $AdS_3$ has a better chance of being consistent if it is based on the coset model. Moreover, it was noticed in [3] that string theory on $AdS_3 \times \mathcal{N}$ is closely related to string theory on $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{U}(1)} \times (\text{time}) \times \mathcal{N}$, the difference lying in the conditions to be satisfied by the zero modes. Therefore it seems important to extend the formalism reviewed in the previous section to the coset theory.

The $\mathrm{SL}(2)/\mathrm{U}(1)$ WZW theory describes string propagation in the background of the two-dimensional black hole [21]. The spectrum of this theory was discussed in [22, 23] and certain correlation functions where computed in [20]. Here we follow a slightly different approach, based on Dotsenko's formulation of the $\mathrm{SU}(2)$ case, which will prove to be useful to manifestly include the spectral flow symmetry into string theory in $AdS_3$.

The procedure to gauge the $\mathrm{U}(1)$ subgroup was introduced in reference [22]. It amounts to adding a new free scalar field $X$ and a $(b, c)$ fermionic ghost system with propagators

$$\langle X(z)X(w)\rangle = -\ln(z-w)\,, \qquad \langle c(z)b(w)\rangle = \frac{1}{z-w}\,. \tag{3.1}$$

We are interested in the euclidean theory, in which the boson $X$ is compact with radius $R=\sqrt{k/2}$. The nilpotent BRST charge of this symmetry is

$$Q^{\mathrm{U}(1)} = \int_{C_0} c\left(J^3 - i\sqrt{\frac{k}{2}}\partial X\right) \tag{3.2}$$

and the stress-energy tensor of the gauged theory is

$$T_{\mathrm{SL}(2)/\,\mathrm{U}(1)} = T_{\mathrm{SL}(2)} - \frac{1}{2}\partial X\partial X - b\partial c\,, \tag{3.3}$$

where $T_{\mathrm{SL}(2)}$ is given in (2.6).

The primary fields of the coset theory should be invariant under $Q^{\mathrm{U}(1)}$. They are given by [20]

$$\Psi^j_m(z) = \gamma^{j-m} e^{\frac{2j}{\alpha_+}\phi} e^{i\sqrt{\frac{2}{k}}mX} \tag{3.4}$$

and their conformal weight is

$$\Delta = -\frac{j(j+1)}{k-2} + \frac{m^2}{k}\,. \tag{3.5}$$

A comment on the antiholomorphic dependence of these fields is in order. In the euclidean black hole theory, the $J_0^3, \bar{J}_0^3$ eigenvalues $m, \bar{m}$ lie on the lattice

$$m = \frac{1}{2}(p+\omega k)\,, \qquad \bar{m} = -\frac{1}{2}(p-\omega k)\,, \tag{3.6}$$

where $p$ (the discrete momentum of the string along the angular direction) and $\omega$ (the winding number) are integers. The sum is $m+\bar{m}=\omega k$ and the difference $m-\bar{m}$ is an integer. This is to be contrasted with the $\widehat{\mathrm{SL}(2,\mathbb{R})}$ case where $m+\bar{m}$ is not quantized. In effect, $m+\bar{m}$ is the spacetime energy of the string in $AdS_3$ and it may take either discrete ($\hat{\mathcal{D}}^{\pm}_j$) or continuous ($\hat{\mathcal{C}}_j$) values.

In order to construct the conformal blocks in the coset theory following the same steps as in the previous section, a secreening operator is needed. It is evident from the expression (3.3) for the stress-energy tensor that the screening operators in the coset theory are the same as in the SL(2) case, namely they are given by equations (2.11).

Next, the operator conjugate to the identity has to be found. Two additional operators to that of the SL(2) theory, $\tilde{\mathcal{I}}_0$ in equation (2.14), exist (they were introduced in reference [3]), namely

$$\tilde{\mathcal{I}}_+ = \gamma^{-k} e^{-\frac{k}{\alpha_+}\phi} e^{i\sqrt{\frac{k}{2}}X}\,, \qquad \tilde{\mathcal{I}}_- = e^{-\frac{k}{\alpha_+}\phi} e^{-i\sqrt{\frac{k}{2}}X}\,. \tag{3.7}$$

It is easy to check that they share the properties of $\tilde{\mathcal{I}}_0$, i.e. they commute with the currents and have vanishing conformal weight. (Actually $J^+(z)\tilde{\mathcal{I}}_+(w)$ and $J^-(z)\tilde{\mathcal{I}}_-(w)$ have a non-vanishing single pole, but a similar argument as the one made for $J^-(z)\tilde{\mathcal{I}}_0(w)$ applies, namely the residues are spurious states which decouple in the conformal blocks).

Correspondingly two new sets of charge asymmetry conditions arise

$$N_\beta - N_\gamma = k \qquad \sum_i \alpha_i = -\frac{k}{\alpha_+} \qquad \sqrt{\frac{2}{k}} \sum_i \xi_i = \sqrt{\frac{k}{2}} \tag{3.8}$$

and

$$N_\beta - N_\gamma = 0 \qquad \sum_i \alpha_i = -\frac{k}{\alpha_+} \qquad \sqrt{\frac{2}{k}} \sum_i \xi_i = -\sqrt{\frac{k}{2}} \,. \tag{3.9}$$

Note that $\tilde{\mathcal{I}}_0$ given by (2.14) is also a good conjugate identity for the coset theory, therefore equations (3.8) and (3.9) should be completed in this case as

$$N_\beta - N_\gamma = k-1 \qquad \sum_i \alpha_i = \frac{2-2k}{\alpha_+} \qquad \sum_i \xi_i = 0\,; \tag{3.10}$$

$\xi_i$ denotes the "*charge*" of the field $X(z_i)$.

Following the procedure outlined in the previous section to find the integral representation of conformal blocks one needs the conjugate representation of the highest weight fields. It is easy to show that the following operators have the correct properties

$$\tilde{\Psi}^{j(0)}_j = \beta^{2j+k-1} e^{-\frac{2(j-1+k)}{\alpha_+}\phi} e^{i\sqrt{\frac{2}{k}}jX} \tag{3.11}$$

and

$$\tilde{\Psi}^{j(-)}_j = \beta^{2j} e^{-\frac{(2j+k)}{\alpha_+}\phi} e^{i\sqrt{\frac{2}{k}}(j-\frac{k}{2})X} \,. \tag{3.12}$$

One can check that the two-point functions $\langle \tilde{\Psi}^{j(0)}_j \Psi^j_{-j}\rangle_0$ and $\langle \tilde{\Psi}^{j(-)}_j \Psi^j_{-j}\rangle_-$ do not require screening operators in order to satisfy equations (3.10) and (3.9) respectively. Correspondingly, the indices (0) and ($-$) refer to the charge asymmetry conditions obtained from the conjugate identities $\tilde{\mathcal{I}}_0$ and $\tilde{\mathcal{I}}_-$. Other conjugate operators in the multiplet $\Psi^j_m$ can be found by acting with $J^-$ on the highest weight conjugate operator. This construction mimics the radial quantization in which the operator $\Psi$ creates one vacuum of the Fock space, and $\tilde{\Psi}$ creates another vacuum, a conjugate one (see reference [7]).

Therefore the $N$-point function in the coset theory takes the form

$$\mathcal{A}^{0,\pm}_N = \left\langle \prod_{i=1}^{N-1} \Psi^{j_i}_{m_i}(z_i) \tilde{\Psi}^{j_N(0),(\pm)}_{j_N}(z_N) \prod_n \mathcal{S}_+(u_n) \prod_m \mathcal{S}_-(v_m) \right\rangle_{0,\pm} \,, \tag{3.13}$$

where the number of screening operators should satisfy the charge asymmetry conditions (3.10), (3.8) or (3.9), the conjugate highest weight operators are defined accordingly and the corresponding amplitudes are denoted by $\mathcal{A}_N^0$, $\mathcal{A}_N^+$ and $\mathcal{A}_N^-$, respectively. It is easy to see that the conjugate operator in the sense of $\tilde{\mathcal{I}}_+$ does not have such a simple form as $\tilde{\Psi}_j^{j(0)}$ or $\tilde{\Psi}_j^{j(-)}$, and thus the corresponding correlator $\mathcal{A}_N^+$ above should be taken as a formal expression.

Notice that the conjugate operators (3.11) and (3.12) create highest weight tachyons and can be used as vertex operators for such states in the intermediate positions $z_2, \ldots, z_{N-1}$, i.e. one can insert up to $N-2$ conjugate operators of any kind in correlation functions, as long as the in- and out- vacuum states are consistent with the scalar product between a direct and a conjugate representation of the Fock space.

At this point it is convenient to recall previous literature. The connection between the free field description of the two dimensional black hole and Liouville theory plus $c = 1$ matter was discussed in [25, 24]. It was shown that, as follows from naive counting of degrees of freedom, the $\beta, \gamma$ system can be decoupled and all the results of this section can be phrased in terms of the $\phi, X$ fields, which describe a $c = 1$ system ($X$) coupled to the Liouville mode ($\phi$). This implies that it might be possible to ignore the problems posed by non-integer powers of the ghost fields and that analytic continuation in the fashion of [18] can be used safely to compute the correlators (3.13).

Correlation functions for string theory in the black hole background were computed in reference [20] by functional integration. The equivalent of the charge asymmetry conditions (3.10) arise in that formalism from the integral of the zero modes of the fields and the background charge in the action functional. In fact, recall that $\Psi_{-j}^{-j-1}$ was used in [20] as the conjugate highest weight operator, $\tilde{\Psi}_j^j$, and the following charge asymmetry conditions were used: $N_\beta - N_\gamma = 1$; $\sum_i \alpha_i = -1$; $\sum_i m_i = 0$.

## 4. Spectral flow and vertex operators of string theory in $AdS_3$

Let us now turn to string theory in $AdS_3$. As noticed by Maldacena and Ooguri [3], the algebra (2.5) has a symmetry given by

$$\begin{aligned} J_n^3 &\to \tilde{J}_n^3 = J_n^3 - \frac{k}{2}\omega\delta_{n,0} \\ J_n^\pm &\to \tilde{J}_n^\pm = J_{n\pm\omega}^\pm \,, \end{aligned} \tag{4.1}$$

where $\omega \in Z$. Consequently, the Laurent coefficients of the stress-energy tensor transform as

$$L_n \to \tilde{L}_n = L_n + \omega J_n^3 - \frac{k}{4}\omega^2\delta_{n,0} \,. \tag{4.2}$$

This is the well known spectral flow symmetry [26]. Classically the parameter $\omega$ represents the winding number of the string around the center of $AdS_3$. Quantum mechanically one can define asymptotic states consisting of long strings whose winding number could in principle change in a scattering process.



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The spectral flow generates new representations of the $SL(2,\mathbb{R})$ algebra. Indeed an eigenstate $|\tilde{j},\tilde{m}\rangle$ of the operators $\tilde{L}_0$ and $\tilde{J}^3$ with the following eigenvalues

$$\begin{aligned}\tilde{L}_0\,|\tilde{j},\tilde{m}\rangle &= -\frac{\tilde{j}(\tilde{j}+1)}{(k-2)}\,|\tilde{j},\tilde{m}\rangle \\ \tilde{J}^3\,|\tilde{j},\tilde{m}\rangle &= \tilde{m}\,|\tilde{j},\tilde{m}\rangle\end{aligned}$$

is also an eigenstate of $L_0$ and $J^3$ with eigenvalues given by

$$\begin{aligned}L_0\,|\tilde{j},\tilde{m}\rangle &= \left(-\frac{\tilde{j}(\tilde{j}+1)}{(k-2)} - \omega\tilde{m} - \frac{k}{4}\omega^2\right)|\tilde{j},\tilde{m}\rangle \\ J^3\,|\tilde{j},\tilde{m}\rangle &= \left(\tilde{m} + \frac{k}{2}\omega\right)|\tilde{j},\tilde{m}\rangle\ .\end{aligned} \tag{4.3}$$

The Hilbert space of string theory in $AdS_3$ can be consequently extended $\mathcal{H} \to \mathcal{H}_\omega$ in order to include the states $|\tilde{j},\tilde{m},\omega\rangle$ obtained by spectral flow, which satisfy the following on-shell condition

$$(L_0-1)\,|\tilde{j},\tilde{m},\omega\rangle = \left(-\frac{\tilde{j}(\tilde{j}+1)}{(k-2)} - \omega\tilde{m} - \frac{k}{4}\omega^2 + L - 1\right)|\tilde{j},\tilde{m},\omega\rangle = 0 \tag{4.4}$$

(compare to equation (2.22) and note that it is now possible to consider bounds on the spin $\tilde{j}$ without limiting the excitation level $L$). The new representations are denoted by $\hat{\mathcal{D}}^{\pm,\omega}_{\tilde{j}}$ and $\hat{\mathcal{C}}^{\omega}_{\tilde{j}}$ and they consist of the spectral flow of the discrete (highest and lowest weight) and continuous series respectively. These representations also contain negative norm states, but Maldacena and Ooguri [3] have shown that restricting the spin $\tilde{j}$ to $\tilde{j} < (k-2)/2$, the Virasoro constraints remove all the ghosts from the theory. Moreover, closure of the spectrum under the spectral flow symmetry implies that the upper unitarity bound on the spin $\tilde{j}$ of the physical states should be $\tilde{j} < (k-3)/2$, i.e. the bound is stronger than required by the no-ghost theorem.

The spectrum of string theory consists then of a product of left and right representations $\hat{\mathcal{C}}^{\omega}_{\tilde{j},L} \times \hat{\mathcal{C}}^{\omega}_{\tilde{j},R}$ and $\hat{\mathcal{D}}^{\pm,\omega}_{\tilde{j},L} \times \hat{\mathcal{D}}^{\pm,\omega}_{\tilde{j},R}$ with the same amount of spectral flow and the same spin $\tilde{j}$ on the holomorphic and antiholomorphic parts and with $-1/2 < \tilde{j} < (k-3)/2$. The partition function containing the spectral flow of the discrete representations with this bound on the spin $\tilde{j}$ was shown to be modular invariant in [3]. Moreover, the partition function for thermal $AdS_3$ backgrounds was also found to be modular invariant and consistent with this spectrum in [30]. From now on we drop the tilde on $\tilde{j},\tilde{m}$.

The spectral flow symmetry has been extensively studied in the context of $N=2$ superconformal field theories. Let us briefly review this case. The $N=2$ superconformal algebra contains in addition to the Virasoro generators $L_n$, two fermionic superpartners $G^{\pm}_n$ and a $U(1)$ current with Laurent coefficients $J_n$. The isomorphism of the algebras generated by $(L_n, G^{\pm}_n, J_n)$ and by the flowed $(\tilde{L}_n, \tilde{G}^{\pm}_n, \tilde{J}_n)$ can be interpreted in terms of the product of some quotient theory whose central charge is

$c-1$ and a free scalar field which bosonizes the U(1) current. Indeed the $N=2$ generators decompose into two mutually commuting sectors, one of which can be expressed in terms of the parafermions defined by Zamolodchikov and Fateev [27] and the other one contains a free boson. This observation led to establish the relation between the $N=2$ discrete series and the representations of SU(2) current algebra. The generalization for $c>3$ was performed in reference [9] by considering the non-compact group $\mathrm{SL}(2,\mathbb{R})$ and the corresponding parafermions introduced in reference [28].

In order to implement this construction in string theory in $AdS_3$ it seems natural to consider the coset $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ (having central charge $c-1$) times a free timelike scalar field $Y(z)$ which bosonizes the $J^3$ current as

$$J^3(z) = -i\sqrt{\frac{k}{2}}\,\partial Y(z) \tag{4.5}$$

and has propagator $\langle Y(z)Y(w)\rangle = \ln(z-w)$. However, instead of the parafermions, one can use the Wakimoto representation introduced in the previous section to describe the coset theory. In this representation the energy-momentum tensor of the full theory takes the form

$$T = \beta\partial\gamma - \frac{1}{2}(\partial\phi)^2 - \frac{1}{\alpha_+}\partial^2\phi - \frac{1}{2}(\partial X)^2 - b\partial c + \frac{1}{2}(\partial Y)^2\,. \tag{4.6}$$

Now a primary field with $J^3$ charge $m$ may be written as

$$V^j_m = \Psi^j_m e^{i\sqrt{\frac{2}{k}}mY(z)}\,. \tag{4.7}$$

where $\Psi^j_m$ is a $J^3$ neutral primary field in the coset theory with conformal weight

$$\Delta(\Psi^j_m) = -\frac{j(j+1)}{k-2} + \frac{m^2}{k}\,. \tag{4.8}$$

In terms of Wakimoto free fields it is possible to write the corresponding vertex operators in the non-compact $\widetilde{\mathrm{SL}(2)}$ case as (see eq. (3.4))

$$V^j_m = \gamma^{j-m} e^{\frac{2j}{\alpha_+}\phi} e^{i\sqrt{\frac{2}{k}}mX} e^{i\sqrt{\frac{2}{k}}mY(z)}\,. \tag{4.9}$$

Now, taking into account the spectral flow, for every field $V^j_m$ in the sector $\omega=0$ one can write a field in the sector twisted by $\omega$ as

$$V^\omega_{j,m} = \gamma^{j-m} e^{\frac{2j}{\alpha_+}\phi} e^{i\sqrt{\frac{2}{k}}mX} e^{i\sqrt{\frac{2}{k}}(m+\omega k/2)Y(z)}\,. \tag{4.10}$$

It has the following conformal weight

$$\Delta(V^\omega_{j,m}) = -\frac{j(j+1)}{k-2} - m\omega - \frac{k\omega^2}{4} \tag{4.11}$$

and therefore it has all the properties to be considered the tachyon vertex operator in the free field representation of string theory in $AdS_3$.



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The general method proposed to construct the theory is then to begin with the local operators that create states of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ and remove the dependence on the boson $X$. Once one has constructed the unitary modules for the coset, one can combine them with the state space of a free boson $Y$ to build in unitary representations of the full string theory on $AdS_3$. Consequently the vertex operators are a direct product of an operator in the $\mathrm{SL}(2)/\mathrm{U}(1)$ coset theory and an operator in the free field sector representing the time direction (note the plus sign in the propagator $\langle Y(z)Y(w)\rangle$).

There seem to be redundant degrees of freedom in this representation. The situation is similar to the description of the two dimensional black hole in terms of Wakimoto free fields plus a free boson. Therefore it may be plausible that a simplified formulation exists also in this case in terms of only three fields [25, 24]. An equivalent expression for the vertex operators in terms of three free fields has been recently introduced in reference [29] in the discrete light-cone parametrization, although a different interpretation is offered. However, both approaches can be shown to be related upon using the constraint that $Q^{\mathrm{U}(1)}$ annihilates the physical states of the coset theory.

Now, in order to complete the formulation of the theory, a prescription to compute correlation functions is needed.

## 5. Scattering amplitudes and factorization

The scattering amplitudes of physical states are essential ingredients to obtain the spectrum and study the unitarity of the theory. Several references have dealt with the problem of computing correlation functions in $H_3^+$ [4, 5, 6] and much progress has been achieved in recent years. But it is difficult to construct higher than three-point functions without making some approximations. It would be interesting to resolve the technical problems in the evaluation of physical correlators as well as to prove unitarity of string theory in $AdS_3$ at the interacting level. In this section we take a step in this direction by extending the construction of the free field representation of correlation functions discussed in the context of the $\mathrm{SL}(2)/\mathrm{U}(1)$ coset in section 3, to string theory in $AdS_3$.

The free field approach is a powerful tool to study the theory near the boundary of spacetime, even though the explicit computation of correlation functions also presents some technical difficulties. In particular, several properties can be obtained from certain limits of the scattering amplitudes. Information about the spectrum is obtained in the limit in which the insertion points of a subset of vertex operators collide to one point. In this region of integration one finds

$$\lim_{z_1,\dots,z_{M-1}\to z_M} \mathcal{A}_N = \sum_{L=0}^{\infty} \frac{\langle V_1 V_2 \cdots V_M \tilde{V}_{J_L}\rangle\langle V_{J_L} V_{M+1}\cdots V_N\rangle}{\Delta(V_{J_L})-1}\,, \tag{5.1}$$

where $\Delta(V_{J_L})$ is the conformal dimension of the vertex operator creating the intermediate state at excitation level $L$. The consistency of the theory can thus be established by starting with unitary external states and analysing the norm of the intermediate states.

In order to implement this factorization process, the simplest starting point is the scattering amplitude of unitary external tachyons. Extending the ideas developed in sections 2 and 3 to the case of string theory in $AdS_3$, the $N$-point functions should take the following form

$$\mathcal{A}_N^{0,\pm} = \left\langle \prod_{i=1}^{N-1} V^{j_i}_{m_i,\omega_i}(z_i) \tilde{V}^{j_N(0),(\pm)}_{j_N,\omega_N}(z_N) \prod_{n=1}^{s} \mathcal{S}_+(u_n) \right\rangle_{0,\pm} , \tag{5.2}$$

where the vertex operators $V^j_{m,\omega}$ are given in equation (4.10) and the conjugate highest weight operators are now

$$\tilde{V}^{j(0)}_{j,\omega} = \beta^{2j+k-1} e^{-\frac{2(j-1+k)}{\alpha_+}\phi} e^{i\sqrt{\frac{2}{k}}jX} e^{i\sqrt{\frac{2}{k}}(j+\frac{k}{2}\omega)Y} \tag{5.3}$$

and

$$\tilde{V}^{j(-)}_{j,\omega} = \beta^{2j} e^{-\frac{(2j+k)}{\alpha_+}\phi} e^{i\sqrt{\frac{2}{k}}(j-\frac{k}{2})X} e^{i\sqrt{\frac{2}{k}}(j+\frac{k}{2}\omega)Y} . \tag{5.4}$$

Non-vanishing correlators require that the number of screening operators satisfy equations (3.8), (3.9) or (3.10) plus an additional charge conservation condition arising from exponentials of the field $Y(z)$, namely

$$\sum \Omega_i = \sum_i \left( m_i + \frac{\omega_i k}{2} \right) = 0 , \tag{5.5}$$

where $\Omega_i$ denotes the "*charge*" of the field $Y(z_i)$.

This is the energy conservation condition. In fact, $m+\bar{m}$ is the total energy of the string in $AdS_3$ which receives kinetic as well as winding contributions. A similar condition arises for the right moving part, and recalling that $\omega = \bar{\omega}$ is implied by periodicity of the closed string coordinates, the left-right matching condition is $\sum_i (m_i - \bar{m}_i) = 0$, i.e. the angular momentum conservation. Consequently, without loss of generality, we shall consider states with the same left and right quantum numbers.

It is interesting to notice that it is possible to construct correlators violating winding number conservation by, for instance, inserting conjugate operators $\tilde{V}^{j(-)}_{j,\omega}$ instead of direct ones into $\mathcal{A}_N^{(0)}$. In fact, correlation functions containing $K$ of these conjugate operators lead to $\sum_i \omega_i = -K$ when combining the last of equations (3.10) with (5.5), whereas processes conserving winding number ($\sum_i \omega_i = 0$) are obtained when inserting direct vertex operators. Recall that it is possible to consider correlators containing up to $N-2$ conjugate operators of a different kind as that required for the conjugate vacuum state, and thus the winding number conservation can be violated by up to $N-2$ units. A similar observation was made in reference [8], where it is argued that in the supersymmetric case the $N$-point functions receive contributions that violate winding number conservation up to $N-2$. Moreover, unpublished work by V. Fateev, A.B. Zamolodchikov and Al.B. Zamolodchikov is quoted, where the same property seems to hold in the bosonic $\mathrm{SL}(2)/\mathrm{U}(1)$ CFT.

In the remaining of this article we shall check the consistency of the formalism introduced in this Section by analysing the factorization properties of the correlators. The procedure is very similar to the one introduced in reference [2] and we include it here for completeness and to stress the differences with the previous construction.

Let us start from $\mathcal{A}_N^{(0)}$, i.e. the $N$-point function for tachyons conserving winding number,

$$\begin{aligned}\mathcal{A}^{(0)j_1\ldots j_N}_{m_1\ldots m_N} &= \frac{1}{\mathrm{Vol}[\mathrm{SL}(2,\mathbb{C})]}\int\prod_{i=1}^{N}d^2z_i\prod_{n=1}^{s}\int d^2w_n\left\langle\prod_{i=1}^{N-1}\gamma_{(z_i)}^{j_i-m_i}\beta_{(z_N)}^{2j_N+k-1}\prod_{n=1}^{s}\beta_{(w_n)}\right\rangle\times\text{c.c.}\\ &\times\left\langle\prod_{i=1}^{N-1}e^{\frac{2}{\alpha_+}j_i\phi(z_i,\bar z_i)}e^{-2\frac{(j_N-1+k)}{\alpha_+}\phi(z_N,\bar z_N)}\prod_{n=1}^{s}e^{-\frac{2}{\alpha_+}\phi(w_n,\bar w_n)}\right\rangle\times\\ &\times\left\langle\prod_{i=1}^{N-1}e^{i\sqrt{\frac{2}{k}}m_iX(z_i,\bar z_i)}e^{i\sqrt{\frac{2}{k}}j_NX(z_N,\bar z_N)}\right\rangle\times\\ &\times\left\langle\prod_{i=1}^{N-1}e^{i\sqrt{\frac{2}{k}}(m_i+\frac{k}{2}\omega_i)Y(z_i,\bar z_i)}e^{i\sqrt{\frac{2}{k}}(j_N+\frac{k}{2}\omega_N)Y(z_N,\bar z_N)}\right\rangle.\end{aligned}$$

Here $(z_i,\bar z_i)$ and $(w_n,\bar w_n)$ are the world-sheet coordinates where the tachyonic and the screening vertex operators, respectively, are inserted. The quantum numbers of the external states and the number of screening operators $s$ have to satisfy equations (3.10) and (5.5). Note that we have taken the direct representation for the vertex operators in intermediate positions $z_2,\ldots,z_{N-1}$. However the discussion below applies equally well (with minor modifications) to cases where one considers some conjugate intermediate vertices.

Using the free field propagators this amplitude becomes

$$\begin{aligned}\mathcal{A}^{(0)j_1\ldots j_N}_{m_1\ldots m_N} &\sim \int\prod_{i=1}^{N}d^2z_i\prod_{r=1}^{s}d^2w_n\ C(z_i,w_n)\bar C(\bar z_i,\bar w_n)\times\\ &\times\prod_{i<j=1}^{N-1}|z_i-z_j|^{-\frac{8j_ij_j}{\alpha_+^2}+\frac{4}{k}m_im_j-\frac{4}{k}(m_i+\frac{k}{2}\omega_i)(m_j+\frac{k}{2}\omega_j)}\\ &\times\prod_{i=1}^{N-1}|z_i-z_N|^{\frac{8j_i(j_N-1+k)}{\alpha_+^2}+\frac{4}{k}m_ij_N-\frac{4}{k}(m_i+\frac{k}{2}\omega_i)(j_N+\frac{k}{2}\omega_N)}\\ &\times\prod_{i=1}^{N-1}\prod_{n=1}^{s}\left(|z_i-w_n|^{8j_i/\alpha_+^2}\,|z_N-w_n|^{-8(j_N-1+k)/\alpha_+^2}\right)\times\prod_{n<m}^{s}|w_n-w_m|^{-8/\alpha_+^2}\end{aligned}\tag{5.6}$$

where $C(z_i,w_n)$ $[\bar C(\bar z_i,\bar w_n)]$ stand for the contribution of the $(\beta,\gamma)$ $[(\bar\beta,\bar\gamma)]$ correlators (see eq. (5.10) below).

Next take the limit $z_2 \to z_1$. The amplitude is expected to exhibit poles on the mass-shell states with residues reproducing the product of 3-point functions times $(N-1)-$point functions. In this particular process there are three equivalent possibilities, namely $\lim_{z_2\to z_1} \mathcal{A}_N^{(0)} \to$

$$\sum_L \frac{\langle V_1 V_2 \tilde{V}^{j_L(0)}_{j_L,\omega_L}\rangle_0 \langle V^{j_L}_{-j_L,-\omega_L} V_3 \cdots \tilde{V}^{(0)}_N\rangle_0}{\Delta(V^{j_L}_{-j_L,-\omega_L})-1} \tag{5.7}$$

$$\sum_L \frac{\langle V_1 V_2 \tilde{V}^{j_L,(+)}_{j_L,\omega_L}\rangle_+ \langle V^{j_L(+)}_{-j_L,-\omega_L} V_3 \cdots \tilde{V}^{(0)}_N\rangle_0}{\Delta(V^{j_L}_{-j_L,-\omega_L})-1} \tag{5.8}$$

$$\sum_L \frac{\langle V_1 V_2 \tilde{V}^{j_L(-)}_{j_L,\omega_L}\rangle_- \langle V^{j_L}_{-j_L,-\omega_L} V_3 \cdots \tilde{V}^{(0)}_N\rangle_0}{\Delta(V^{j_L}_{-j_L,-\omega_L})-1} \tag{5.9}$$

where the subindices refer to the different charge asymmetry conditions, i.e. the number of screening operators whose insertion points are taken in the limit $w_n \to z_1$ in order to produce non-vanishing 3$-$point functions verifies equations (3.10), (3.8) or (3.9), respectively (and the corresponding conjugate operator has to be considered), and the remaining screenings in the $(N-1)-$point functions verify conditions (3.10). $\Delta(V^{j_L}_{j_L,\omega_L})$ refer to the conformal dimensions of the intermediate states at excitation level $L$. Even though we have explicitly constructed the correlators using the conjugate operator in the highest weight position, the quantum numbers of the intermediate states can be general (i.e. not necessarily $j=m$).

To isolate the singularities arising in the intermediate channels perform the change of variables: $z_1 - z_2 = \varepsilon$, $z_1 - v_n = \varepsilon y_n$, $v_n - z_2 = \varepsilon(1-y_n)$, where we have renamed as $v_n$ the insertion points of the $s_1$ screening operators that are necessary to produce non-vanishing 3$-$point functions. In order to extract the explicit $\varepsilon$ dependence of the amplitude it is convenient to write the contribution of the $(\beta,\gamma)$ system as

$$\begin{aligned}
C(z_i, v_n, w_m) &= \left\langle \gamma^{j_1-m_1}_{(z_1)} \gamma^{j_2-m_2}_{(z_2)} \prod_{i=3}^{N-1} \gamma^{j_i-m_i}_{(z_i)} \beta^{2j_N+k-1}_{(z_N)} \prod_{n=1}^{s_1} \beta_{(v_n)} \prod_{m=1}^{s_2} \beta_{(w_m)} \right\rangle \\
&\sim \sum_{\mathrm{Perm}(v_n)} \sum_{r=0}^{s_1} \frac{(j_1-m_1)(j_1-m_1-1)\cdots(j_1-m_1-r+1)}{(z_1-v_1)(z_1-v_2)\cdots(z_1-v_r)} \times \\
&\quad \times \frac{(j_2-m_2)(j_2-m_2-1)\cdots(j_2-m_2-s_1+r+1)}{(z_2-v_{r+1})\cdots(z_2-v_{s_1})} \times \\
&\quad \times \left\langle \gamma^{j_1-m_1-r}_{(z_1)} \gamma^{j_2-m_2-s_1+r}_{(z_2)} \prod_{i=3}^{N-1} \gamma^{j_i-m_i}_{(z_i)} \beta^{2j_N+k-1}_{(z_N)} \prod_{m=1}^{s_2} \beta_{(w_m)} \right\rangle + \\
&\quad + \sum_{\mathrm{Perm}(v_n)} \sum_{r=0}^{s_1-1} \frac{(j_1-m_1)(j_1-m_1-1)\cdots(j_1-m_1-r+1)}{(z_1-v_1)(z_1-v_2)\cdots(z_1-v_r)} \times
\end{aligned} \tag{5.10}$$

$$\times \frac{(j_2-m_2)(j_2-m_2-1)..(j_2-m_2-s_1+r+2)}{(z_2-v_{r+1})\cdots(z_2-v_{s_1-1})} \sum_{i=3}^{N} \frac{j_i-m_i}{z_i-v_{s_1}} \times$$
$$\times \left\langle \gamma_{(z_1)}^{j_1-m_1-r} \gamma_{(z_2)}^{j_2-m_2-s_1+r+1} \prod_{l\neq i} \gamma_{(z_l)}^{j_l-m_l} \gamma_{(z_i)}^{j_i-m_i-1} \beta_{(z_N)}^{2j_N+k-1} \prod_{m=1}^{s_2} \beta_{(w_m)} \right\rangle + \cdots$$

and similarly for $\bar{C}(\bar{z}_i, \bar{v}_n, \bar{w}_n)$. The sign $\sim$ stands for an irrelevant phase. The products $(j_1-m_1)(j_1-m_1-1)\cdots(j_1-m_1-r+1)$ have to be understood as not contributing for $r=0$ (similarly $(j_2-m_2)\cdots(j_2-m_2-s_1+r+1)$ for $r=s_1$). The dots stand for lower order contractions between the fields inserted at $z_1$ and $z_2$ and the $s_1$ screening operators. Note that these functions can be written as a power series in $\varepsilon$ after performing the change of variables and extracting the leading $\varepsilon^{-s_1}$ divergence. Note that these expressions are obtained by treating the powers of the $\beta, \gamma$ fields as positive integers and assuming that analytic continuation can be safely performed.

The amplitude becomes then formally in the limit

$$\mathcal{A}^{(0)j_1..j_N}_{m_1...m_N} \sim \int d^2\varepsilon\, |\varepsilon|^{2s_1-\frac{1}{\alpha_+^2}[8j_1j_2-8s_1(j_1+j_2)+4s_1(s_1-1)]-2(m_1\omega_2+m_2\omega_1+\omega_1\omega_2\frac{k}{2})-2s_1} \times \tag{5.11}$$
$$\times \int d^2z_1 \prod_{i=3}^{N} \int d^2z_i \prod_{n=1}^{s_2} \int d^2w_n \prod_{r=1}^{s_1} \int d^2y_r\, |\Phi(\varepsilon, z_1, z_i, y_r, w_n)\Psi(z_i, w_n)|^2 .$$

The first term in the exponent of $|\varepsilon|$ comes from the change of variables in the insertion points of the $s_1$ screening operators whereas the last term cancelling it arises in the $\beta-\gamma$ system. The other terms in the exponent originate in the contractions of the exponentials. The function $\Phi$ is a regular function in the limit $\varepsilon \to 0$. It is convenient to write separately the contribution to $\Phi$ from the exponentials ($E$) and from the $\beta-\gamma$ system ($C$), i.e. $\Phi = E \times |C|^2$, where $E(\varepsilon, z_1, z_i, y_r, w_n)$ is

$$\begin{aligned}
E = & \prod_{r=1}^{s_1} |y_r|^{8j_1/\alpha_+^2} |1-y_r|^{8j_2/\alpha_+^2} \prod_{r<t} |y_r-y_t|^{-8/\alpha_+^2} \times \\
& \times \prod_{i=3}^{N-1} |z_1-z_i|^{-8j_1j_i/\alpha_+^2-2m_1\omega_i-2\omega_1m_i-k\omega_1\omega_i} |z_1-\varepsilon-z_i|^{-8j_2j_i/\alpha_+^2-2m_2\omega_i-2\omega_2m_i-k\omega_2\omega_i} \times \\
& \times |z_1-z_N|^{8j_1(j_N-1+k)/\alpha_+^2-2(m_1\omega_N+\omega_1j_N)-k\omega_1\omega_N} \\
& \times |z_1-\varepsilon-z_N|^{8j_2(j_N-1+k)/\alpha_+^2-2(m_2\omega_N+\omega_2j_N)-k\omega_2\omega_N} \times \\
& \times \prod_{m=1}^{s_2} |z_1-w_m|^{8j_1/\alpha_+^2} |z_1-\varepsilon-w_m|^{8j_2/\alpha_+^2} \times \\
& \times \prod_{i=3}^{N-1} \prod_{r=1}^{s_1} |z_i-z_1+\varepsilon y_r|^{8j_i/\alpha_+^2} \prod_{r=1}^{s_1} \prod_{m=1}^{s_2} |z_1-\varepsilon y_r-w_m|^{-8/\alpha_+^2} \times \\
& \times \prod_{r=1}^{s_1} |z_N-z_1+\varepsilon y_r|^{-8(j_N-1+k)/\alpha_+^2}
\end{aligned} \tag{5.12}$$

and

$$
\begin{aligned}
C(\varepsilon, z_1, z_i, y_r, w_n) \sim & \sum_{\mathrm{Perm}(y_n)} \sum_{r=0}^{s_1} \frac{(j_1-m_1)(j_1-m_1-1)\cdots(j_1-m_1-r+1)}{y_1 y_2 \cdots y_r} \times \\
& \times \frac{(j_2-m_2)(j_2-m_2-1)\cdots(j_2-m_2-s_1+r+1)}{(1-y_{r+1})(1-y_{r+2})\cdots(1-y_{s_1})} \times \\
& \times \sum_{\mathrm{Perm}(w_m)} \left\{ \left[ -\frac{(j_1-m_1-r)}{z_1-w_1} - \frac{(j_2-m_2-s_1+r)}{(z_1-\varepsilon-w_1)} \right] \times \right. \\
& \qquad \times \left\langle \prod_{i=3}^{N-1} \gamma_{(z_i)}^{j_i-m_i} \beta_{(z_N)}^{2j_N+k-1} \prod_{m=2}^{s_2} \beta_{(w_m)} \right\rangle + \\
& \qquad + \left[ \frac{(j_1-m_1-r)(j_1-m_1-r-1)}{(z_1-w_1)(z_1-w_2)} + \right. \\
& \qquad\quad + \frac{(j_2-m_2-s_1+r)(j_2-m_2-s_1+r-1)}{(z_1-\varepsilon-w_1)(z_1-\varepsilon-w_2)} + \\
& \qquad\quad + \frac{(j_1-m_1-r)(j_2-m_2-s_1+r)}{(z_1-w_1)(z_1-\varepsilon-w_2)} + \\
& \qquad\quad \left. + \frac{(j_1-m_1-r)(j_2-m_2-s_1+r)}{(z_1-w_2)(z_1-\varepsilon-w_1)} \right] + \\
& \qquad + \left. \left\langle \prod_{i=3}^{N-1} \gamma^{j_i-m_i}(z_i) \beta^{2j_N+k-1}(z_N) \prod_{m=3}^{s_2} \beta(w_m) \right\rangle + \cdots \right\} + \cdots
\end{aligned}
\tag{5.13}
$$

The dots inside the bracket in the last equation stand for terms involving more contractions among the vertices at $z_1$ and $z_2$ and the vertex operator at $z_N$ or the $s_2$ screenings at $w_m$, whereas the dots at the end stand for lower order contractions between the colliding vertices ($V^{\omega_1}_{(j_1,m_1)}$ and $V^{\omega_2}_{(j_2,m_2)}$) and the $s_1$ screenings at $v_n$.

The function $\Psi$ in eq. (5.11) is independent of $\varepsilon$.

It is possible to Laurent expand $\Phi$ as

$$
\Phi = \sum_{n,m,l,\bar{l}} \frac{1}{n!m!} \varepsilon^{n+l} \bar{\varepsilon}^{m+\bar{l}} \partial^n \bar{\partial}^m \Phi_{l\bar{l}|\varepsilon=\bar{\varepsilon}=0} \,,
$$

where $\Phi_{l\bar{l}}$ denotes the contributions from terms in $C(z_i, v_n, w_m)$ and $\bar{C}(\bar{z}_i, \bar{v}_n, \bar{w}_m)$ where a number $l$ ($\bar{l}$) of the $\beta$-fields ($\bar{\beta}$) in the $s_1$ screenings are not contracted with the $\gamma$-fields ($\bar{\gamma}$) of the vertices at $z_1$ and $z_2$, but with the other vertices at $z_i$, $i = 3, \ldots, N-1$.

Inserting this expansion in (5.11) and performing the integral over $\varepsilon$, the result is

$$
\begin{aligned}
& \mathcal{A}^{(0)j_1 \ldots j_N}_{m_1 \ldots m_N} \sim \\
& \sim \sum_{n,l} \frac{\Lambda^{[-8j_1j_2+8s_1(j_1+j_2)-4s_1(s_1-1)]/\alpha_+^2 - 2(m_1\omega_2+m_2\omega_1+\omega_1\omega_2 k/2)+2n+2l+2}}{-\frac{8}{\alpha_+^2} j_1 j_2 + \frac{8}{\alpha_+^2} s_1(j_1+j_2) - \frac{4}{\alpha_+^2} s_1(s_1-1) - 2(m_1\omega_2 + m_2\omega_1 + \omega_1\omega_2 \frac{k}{2}) + 2n + 2l + 2}
\end{aligned}
$$

$$\times \int d^2 z_1 \prod_{t=1}^{s_2} \int d^2 w_t \prod_{i=3}^{N} \int d^2 z_i \prod_{r=1}^{s_1} \int d^2 y_r \frac{1}{(n!)^2} \partial^n \Phi_{l|\varepsilon=0} \Psi(z_i, w_t) \times \text{c.c.}\,, \tag{5.14}$$

where $\Lambda$ is an infrared cut-off, irrelevant on the poles.

Let us analyse the pole structure of this expression, namely

$$-\frac{4}{\alpha_+^2} j_1 j_2 + \frac{4}{\alpha_+^2} s_1 (j_1 + j_2) - \frac{2}{\alpha_+^2} s_1 (s_1 - 1) - \left( m_1 \omega_2 + m_2 \omega_1 + \omega_1 \omega_2 \frac{k}{2} \right) + 1 + n + l = 0\,. \tag{5.15}$$

This is precisely the mass shell condition for a highest weight state at level $L = n + l$ with $j = j_1 + j_2 - s_1$, $m = m_1 + m_2$ and $\omega = \omega_1 + \omega_2$, i.e.

$$-\frac{2}{\alpha_+^2} j(j+1) - m\omega - \frac{k}{4}\omega^2 + L = 1 \tag{5.16}$$

if $j_1, m_1, \omega_1$ and $j_2, m_2, \omega_2$ are the quantum numbers of the external on-mass-shell tachyons (namely, $-2j_1(j_1+1)/\alpha_+^2 - m_1\omega_1 - \frac{k}{4}\omega_1^2 = 1$).

At this point it is important to recall that the scattering amplitudes (and the charge asymmetry conditions) are constructed including one conjugate highest weight field. In general, the conjugate vertex operator is a complicated expression, except for the highest weight state and this is the reason why these particular correlators are considered. Therefore, the consistency of the factorization procedure, i.e. the non trivial fact that the number of screening operators contained in the original amplitude can be split in two parts giving rise exactly to two non-vanishing correlators in the residues, has been checked explicitly in the special case in which the intermediate states verify $j = m$. However, more general processes might be considered (i.e. not necessarily containing highest weight states).

Next, let us consider the residues. At lowest order ($n = l = 0$), the amplitude $\mathcal{A}^{(0) j_1 \ldots j_N}_{m_1 \ldots m_N}(\varepsilon = 0)$ reads

$$\begin{aligned}
&\prod_{r=1}^{s_1} \int d^2 y_r \prod_{r=1}^{s_1} |y_r|^{8j_1/\alpha_+^2} |1 - y_r|^{8j_2/\alpha_+^2} \prod_{r<t}^{s_1} |y_r - y_t|^{-8/\alpha_+^2} \times C'(y_r)\bar{C}'(\bar{y}_r) \times \\
&\times \int d^2 z_1 \int \prod_{i=3}^{N} d^2 z_i \prod_{n=1}^{s_2} \int d^2 w_n \, |z_1 - z_i|^{-8(j_1+j_2-s_1)j_i/\alpha_+^2 - 2(m_1+m_2)\omega_i - (\omega_1+\omega_2)(2m_i + k\omega_i)} \times \\
&\times \prod_{n=1}^{s_2} |z_1 - w_n|^{8(j_1+j_2-s_1)/\alpha_+^2} \prod_{3<i<k}^{N-1} |z_i - z_k|^{-8j_i j_k/\alpha_+^2} \prod_{i=3}^{N-1} \prod_{n=1}^{s_2} |z_i - w_n|^{8j_i/\alpha_+^2} \times \\
&\times \prod_{i=1}^{N-1} |z_i - z_N|^{\frac{8j_i(j_N - 1 + k)}{\alpha_+^2} + \frac{4}{k} m_i j_N - \frac{4}{k}(m_i + \frac{k}{2}\omega_i)(j_N + \frac{k}{2}\omega_N)} \prod_{n=1}^{s_2} |z_N - w_n|^{-8(j_N - 1 + k)/\alpha_+^2} \times \\
&\times \prod_{n<m}^{s_2} |w_n - w_m|^{-8/\alpha_+^2} \times C''(z_1, z_i, w_n)\bar{C}''(\bar{z}_1, \bar{z}_i, \bar{w}_n)\,,
\end{aligned} \tag{5.17}$$

where

$$C'(y_r) = \sum_{\text{Perm}(y_n)} \sum_{r=0}^{s_1} \frac{(j_1 - m_1)(j_1 - m_1 - 1)\cdots(j_1 - m_1 - r + 1)}{y_1 y_2 \cdots y_r} \times$$
$$\times \frac{(j_2 - m_2)(j_2 - m_2 - 1)\cdots(j_2 - m_2 - s_1 + r + 1)}{(1 - y_{r+1})(1 - y_{r+2})\cdots(1 - y_{s_1})}$$

and clearly from eq. (5.13) evaluated at $\varepsilon = 0$,

$$C''(z_1, z_i, w_n) = \left\langle \gamma_{(z_1)}^{j_1 - m_1 + j_2 - m_2 - s_1} \prod_{i=3}^{N-1} \gamma_{(z_i)}^{j_i - m_i} \beta_{(z_N)}^{2j_N + k - 1} \prod_{m=1}^{s_2} \beta_{(w_m)} \right\rangle .$$

This can be easily interpreted as the product of a 3-tachyon amplitude (the first line in expression (5.17))

$$\left\langle V_{j_1,m_1}^{\omega_1}(0) V_{j_2,m_2}^{\omega_2}(1) \tilde{V}_{j,m}^{\omega}(\infty) \prod_{r=1}^{s_1} \mathcal{S}_+(y_r) \right\rangle \tag{5.18}$$

times a $(N-1)$-tachyon amplitude

$$\left\langle \gamma_{(z_1)}^{(j_1+j_2-s_1)-m_1-m_2} \gamma_{(z_3)}^{j_3-m_3} \cdots \beta_{(z_N)}^{2j_N+k-1} \prod_{n=1}^{s_2} \beta_{(w_n)} \right\rangle \times \tag{5.19}$$

$$\times \left\langle \bar{\gamma}_{(\bar{z}_1)}^{(j_1+j_2-s_1)-m_1-m_2} \bar{\gamma}_{(\bar{z}_3)}^{j_3-m_3} \cdots \bar{\beta}_{(\bar{z}_N)}^{2j_N+k-1} \prod_{r=1}^{s_2} \bar{\beta}_{(\bar{w}_n)} \right\rangle \times$$
$$\times \left\langle e^{2(j_1+j_2-s_1)\phi(z_1,\bar{z}_1)/\alpha_+} \prod_{i=3}^{N-1} e^{2j_i\phi(z_i,\bar{z}_i)/\alpha_+} e^{-2(j_N-1+k)\phi(z_N,\bar{z}_N)/\alpha_+} \prod_{n=1}^{s_2} e^{-2\phi(w_n,\bar{w}_n)/\alpha_+} \right\rangle \times$$
$$\times \left\langle e^{i\sqrt{\frac{2}{k}}(m_1+m_2)X(z_1,\bar{z}_1)} \prod_{i=3}^{N-1} e^{i\sqrt{\frac{2}{k}}m_i X(z_i,\bar{z}_i)} e^{i\sqrt{\frac{2}{k}}j_N X(z_N,\bar{z}_N)} \right\rangle \times$$
$$\times \left\langle e^{i\sqrt{\frac{2}{k}}(m_1+m_2+(\omega_1+\omega_2)\frac{k}{2})Y(z_1,\bar{z}_1)} \prod_{i=3}^{N-1} e^{i\sqrt{\frac{2}{k}}(m_i+\frac{k}{2}\omega_i)Y(z_i,\bar{z}_i)} e^{i\sqrt{\frac{2}{k}}(j_N+\frac{k}{2}\omega_N)} \right\rangle . \tag{5.20}$$

Therefore, the tachyon vertex operator can be reconstructed, namely

$$V_{(j,m)}^{\omega}(z,\bar{z}) = \gamma^{j-m}(z)\bar{\gamma}^{j-m}(\bar{z}) e^{\frac{2}{\alpha_+}j\phi(z,\bar{z})} e^{i\sqrt{\frac{2}{k}}mX(z,\bar{z})} e^{i\sqrt{\frac{2}{k}}(m+\frac{k}{2}\omega)Y(z,\bar{z})} . \tag{5.21}$$

with $j = j_1 + j_2 - s_1$, $m = m_1 + m_2$ and $\omega = \omega_1 + \omega_2$.

The vertex operators creating states at higher excitation levels can be obtained from the higher order terms in the Laurent expansion (5.14) following the same steps implemented in this section.

## 6. Conclusions and discussion

The near boundary limit of string theory in $AdS_3$ has been considered using the Wakimoto free field representation of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. The theory was taken as a direct product of the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset and a timelike free boson. The winding sectors obtained by the spectral flow transformation appear naturally in the spectrum of the theory. Correlation functions of physical states were constructed extending to the non-compact case, Dotsenko's integral representation of conformal blocks in the SU(2) case. There are three sets of charge asymmetry conditions arising from the corresponding conjugate identity operators. Conjugate vertex operators can be constructed with the help of these conditions, and they can be used to describe scattering processes either conserving or violating winding number (in the latter case by up to $N-2$ units). We have explicitly constructed these conjugate operators for the highest weight states and indicated the procedure that should be followed to find more general conjugate operators. The consistency of the formalism was checked in the factorization limit obtained when the insertion points of two external vertex operators coincide on the sphere. In this limit the amplitudes were shown to exhibit poles on the mass-shell states and residues reproducing the products of $3-$ and $(N-1)-$point functions of the external states with the intermediate on-mass shell states.

This formalism can be used to compute scattering amplitudes and to study the unitarity of string theory in $AdS_3$ at the interacting level, similarly as was done in reference [2]. In fact, starting from the scattering amplitudes of unitary external states, one can analyse the quantum numbers of the intermediate states and determine if they fall within the unitarity bound. However the procedure has some limitations and more work is necessary to achieve this goal, as well as to explicitly evaluate the correlators.

The radial-type quantization that we have considered requires a conjugate field defining a conjugate vacuum state. The vertex operator for such field is in general a complicated expression, unless it is a highest weight field. In order to find unitary highest weight tachyons in string theory one has to take into account an internal compact space, i.e. string theory on $AdS_3 \times \mathcal{N}$ has to be considered, and this requires working out an explicit example.

On the other hand, if one is interested in applications to string theory, the asymptotic states naturally leading to the definition of the S-matrix consist of long strings. The states describing the long strings belong to the spectral flow of the continuous representation [3] while the correlators discussed in this article contain at least one highest weight field. It is usually assumed that an analytical continuation can be performed and that both real or complex values of the spin $j$ can be treated on an equal footing. But an explicit calculation of mixed fusion rules (involving states both in the continuous and discrete representations) requires the functional form of the conjugate operators for all representations, including the continuous series. Moreover there is no clear physical interpretation of the role played by the discrete representations in the scattering amplitudes of asymptotic string states. The extension of the objects constructed in this article to take into account states of the continuous representation is necessary for a complete understanding of the theory.

Another aspect of the construction which requires more thought is the absence of simple conjugate highest weight operators with respect to the charge asymmetry conditions (3.8). It would be important to have explicit expressions for these operators since it is likely that they can be used to describe scattering amplitudes violating winding number conservation by a positive integer.

Explicit evaluation of 3−point functions with this formalism would be important in order to compare with other approaches [5, 6]. Furthermore the extension of the construction presented here to superstring theory should be interesting.


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